**Numerical Methods For Science and Engineering**

**Lecture Note 2**

**Solution of Linear System**

**2.1 Solution of Linear System by Elimination**

**Equivalent Systems:** Two systems of equations are called equivalent if and only if they have the same solution set.

**Elementary Transformations:** A system of equations is transformed into an equivalent system if the following elementary operations are applied on the system:

(1) two equations are interchanged

(2) an equation is multiplied by a non-zero constant

(3) an equation is replaced by the sum of that equation and a multiple of any other equation.

**Gaussian Elimination:** The process which eliminates an unknown from succeeding equations using elementary operations is known as Gaussian elimination.

The equation which is used to eliminate an unknown from the succeeding equations is known as the **pivotal equation**. The coefficient of the eliminated variable in a pivotal equation is known as the **pivot**. If the pivot is zero, it cannot be used to eliminate the variable from the other equations. However, we can continue the elimination process by interchanging the equation with a nonzero pivot.

**Solution of a Linear System**

A systematic procedure for solving a linear system is to reduce a system that is easier to solve. One such system is the echelon form. The **back substitution** is then used to solve the system in reverse order.

A system is in ***echelon form*** or ***upper triangular form*** if

(i) all equations containing nonzero terms are above any equation with zeros only.

(ii) the first nonzero term in every equation occurs to the right of the first nonzero term term in the equation above it.

**Solution of a Linear System**

A systematic procedure for solving a linear system is to reduce a system that is easier to solve. One such system is the upper triangular form. The **back substitution** is then used to solve the system in reverse order.

**2.2 Pivotal Elimination Method**

Computers and calculators use fixed number of digits in its calculation and we may need to round the numbers. This introduces error in the calculations. Also, when two nearly equal numbers are subtracted, the accuracy in the calculation is lost. Consider the following example to study the difficulty faced in the elimination method.

**Example 2.1 :**

The linear system





has the exact solution  and .

Solve the above system using four-digit arithmetic i.e. keeping 4 s.f. at each value by

(a) Gaussian elimination without changing the order.

(b) Gaussian elimination by interchanging the order of the equations.

**Solution**

Using four-digit arithmetic the system can be written as

 (E1)

 (E2)

(a) Here a multiplier is .

Multiplying (E1) by 1410 and retaining 4 digits with rounding, we have





Subtracting we get



To 4 digits this can be written as



This gives



Substituting the value of *y* in (E1), we get





This produces a wrong result.

(b) Interchanging the rows, we can write

 (E3)

 (E4)

For this system multiplier is



The replacement of (E4) by  reduces the system t0





The 4-digits arithmetic gives the solution

 and 

which is the exact solution of the system.

This example shows that the order in which we treat the equations for elimination affects the accuracy in the elimination process.

Note that . Thus in eliminating *x*, we should use the equation with maximum numerical coefficient of *x*.

**2.2.1 Gaussian Elimination with Partial Pivoting**

(a) Select pivotal equation for a variable for elimination (the equation with maximum numerical coefficient of that variable in the system).

(b) Eliminate the chosen variable from the remaining equations with respect to the pivotal equation.

(c) Repeat the process for the subsystem.

(d) Using back-substitution find solutions by using pivotal equations.

For convenience of calculations we made the coefficients of eliminated variable to unity before elimination. This is illustrated through an example below.

**Example 2.2 :** The currents running through an electrical [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) are given by the following [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) of equations.  The three currents, I1, I2, and I3, are measured in amps.

.

a. Solve the above [system](javascript:def('/Glossary/glossaryterm.aspx?word=System',%20500,%20500);) to find the currents in this circuit using Gaussian elimination with partial pivoting.

b. Justify your result by direct substitution in the original equation.

c. Write MATLAB codes to solve by left division (backslash) operator.

**Solution :**  a.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Oper |  |  |  | Constant | Eqn # |  |
|  | 8 | 3 | -5 | 3 | E1 |  |
|  | 10 | 7 | 2 | 4 | E2 |  |
|  | 6 | 4 | 7 | 8 | E3 |  |
| E2/10 | 1.000 | 0.700 | 0.200 | 0.400 | E4 | 1st Piv eqn |
| E1/8 | 1.000 | 0.375 | -0.625 | 0.375 | E5 |  |
| E3/6 | 1.000 | 0.667 | 1.167 | 1.333 | E6 |  |
| E5-E4 | 0 | -0.325 | -0.825 | -0.025 | E7 |  |
| E6-E4 | 0 | -0.033 | 0.967 | 0.933 | E8 |  |
| E7/(-0.325) | 0 | 1 | 2.538 | 0.0769 | E9 | 2nd Piv eqn |
| E8/(-0.03) | 0 | 1 | -32.233 | -31.1 | E10 |  |
| E9-E10 | 0 | 0 | 34.771 | 31.177 |  |  |
| E10/(-32.233) | 0 | 0 | 1 | 0.897 |  | 3rd Piv eqn |

The triangular form of the system can be summarized as

Solving by back substitution, we have,

b. Justification:

c. >> A=[8 3 -5; 10 7 2; 6 4 7];

>> b=[3; 4; 8];

>> Solution = A\b

Solution = 1.7500, -2.1829, 0.8902

**Example 2.3 :** Cory, Josh and Dan went shopping for Halloween treats. Cory bought 3 chocolate

pumpkins, 4 masks and 8 candy witches. He spent $36.65. Josh bought 5 chocolate pumpkins,

3 masks and 10 candy witches. He spent $37.50. Dan bought 4 chocolate pumpkins, 5 masks

and 6 candy witches. He spent $43.45.

i. Write a system of equations to represent this problem

ii. Calculate the unit price of each item purchased using Gaussian elimination with pivoting.

**Solution:**

i. Let price of a chocolate pumpkin

price of a mask

price of a candy witche

Now we can setup our system of equations as:

ii.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Oper |  |  |  | Constant | Eqn # |  |
|  | 3 | 4 | 8 | 36.65 | E1 |  |
|  | 5 | 3 | 10 | 37.50 | E2 |  |
|  | 4 | 5 | 6 | 43.45 | E3 |  |
| E2/5 | 1.000 | 0.600 | 2.000 | 7.500 | E4 | 1st Piv eqn |
| E1/3 | 1.000 | 1.333 | 2.667 | 12.217 | E5 |  |
| E3/4 | 1.000 | 1.250 | 1.500 | 10.863 | E6 |  |
| E4-E5 | 0 | -0.733 | -0.667 | -4.717 | E7 |  |
| E4-E6 | 0 | -0.650 | 0.500 | -3.363 | E8 |  |
| E7/(-0.733) | 0 | 1.000 | 0.910 | 6.435 | E9 | 2nd Piv eqn |
| E8/(-0.650) | 0 | 1.000 | -0.769 | 5.174 | E10 |  |
| E9-E10 | 0 | 0 | 1.679 | 1.261 | E11 |  |
| E10/(1.679) | 0 | 0 | 1.000 | 0.751 | E12 | 3rd Piv eqn |

The triangular form of the system can be summarized as

Solving by back substitution, we have,

Partial pivoting is adequate for most of the simultaneous equations which arise in practice. But we may encounter sets of equations where wrong or incorrect solutions may results. To improve the calculation in such cases **total pivoting** is used. In total pivoting, maximum magnitude of the coefficients of the variables is used for the pivot in each elimination.

**2.3 Solution of Linear System by Iterative Method**

Iterative method for linear system is similar as the method of fixed-point iteration for an equation in one variable. To solve a linear system by iteration, we solve each equation for one of the variables, in turn, in terms of the other variables. Starting from an approximation to the solution, if convergent, derive a new sequence of approximations. Repeat the calculations till the required accuracy is obtained.

An iterative method converges, for any choice of the first approximation, if every equation satisfies the condition that the magnitude of the coefficient of solving variable is greater than the sum of the absolute values of the coefficients of the other variables. A system satisfying this condition is called diagonally dominant. A linear system can always be reduced to diagonally dominant by elementary operations.

For example, in the system  (E1)  

 (E2) 

 (E3) 

is not diagonally dominant. Rearranging as (E3) − (E1), (E2), (E1), we have

 

 



The system reduces to diagonally dominant form.

**2.3.1. Jacobi Iterative Method:**

In this method, a fixed set of values is used to calculate all the variables and then repeated for the next iteration with the values obtained previously. The iterative formulas of the above system are

Gauss-Seidel iterative method is more efficient than Jacobi’s iterative method and explained through an example.

**2.3.2. Gauss-Seidel Iterative Method**

In this method, the values of each variable is calculated using the most recent approximations to the values of the other variables. The Gauss-Seidel iterative formulas of the above system are

If initial values are not supplied we can start with initial values

and perform the iterations until required accuracy is achieved.

**Example 2.4:** Given the linear system .

a. Reduce the above system to diagonally dominant form.

b. Write the corresponding Gauss-Seidel iteration formula.

c. Compute two iterations to estimate the roots to 2 d.p. with and .

d. Write MATLAB code to iterate the above formula four times.

**Solution:**

1. Eq(2)

Eq(1)-Eq(2)

Eq(2)-Eq(3)

1. Gauss-Seidel formula

1. Starting with initial values

When , we have

For , we have

Solution to 2 d.p. is

d. >> x(1)=1.5;

>> y(1)=-1;

>> z(1)=1;

>> iter(1)=0;

>> for n=1:4

iter(n+1)=n;

x(n+1)=(16+3\*y(n)-2\*z(n))/8;

y(n+1)=(-9+2\*x(n+1)-z(n))/8;

z(n+1)=(1+2\*x(n+1)-4\*y(n+1))/10;

end

>> Solution = [iter',x',y',z']

Solution =

0 1.5000 -1.0000 1.0000

1.0000 1.3750 -0.9063 0.7375

2.0000 1.4758 -0.8482 0.7345

3.0000 1.4983 -0.8422 0.7366

4.0000 1.5000 -0.8421 0.7368

**Exercise-2**

|  |  |  |
| --- | --- | --- |
| **1.** | A chemist has 3 different acid solutions. The first acid solution contains acid, the second contains and the third contains acid. The chemist wants to use all 3 solutions to obtain a mixture of 100 liters containing acid. The supply of acid is low. Consequently, twice as much solution as solution must be used. How many liters of each solution should be used? | |
|  |  | |
| **2.** | A movie theater is filled the capacity of The theater charges taka for children, 70 for students and 120 for adults. There are half as many adults as there are students. If the total ticket sales were 24150 taka, how many children, students and adults attended? | |
|  |  |  |
| **3.** |  |  |
|  | i. | Solve the above linear system using Gaussian elimination with pivoting (partial/total). |
|  | ii. | Justify your result by direct substitution in the original equations. |
|  | iii. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **4.** |  | using    using  using |
|  | i. | Reduce the above system to diagonally dominant form. |
|  | ii. | Write the corresponding Gauss-Seidel and Jacobi iteration formula. |
|  | iii. | Compute two iterations to estimate the roots to 3 d.p. with the given initial values. |
|  | iv. | Justify your result by direct substitution in the original equations. |
|  | v. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **5.** | Consider the linear system: | |
|  | i. | Reduce the above system to diagonally dominant form. |
|  | ii. | Write the corresponding Gauss-Seidel iteration formula. |
|  | iii. | Compute two iterations to estimate the roots to 2 d.p with the following initial values *x=* 2, *y*=-0.75, *z=*-0.2. |
|  | iv. | Justify your result by direct substitution in the original equations. |
|  | v. | Write MATLAB codes to iterate the above formula four times. |
|  |  |  |
| **6.** | Determine the loop currents of the given circuit by solving the linear system using  Gaussian elimination with pivoting.   |  |  | | --- | --- | | For loop 1:  For loop 2:  For loop 3: | Description: C:\Users\Administrator\Desktop\circuit2.tif | | |
|  |  |  |
| **7.** | Given the linear system: | |
|  | i. | Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p. |
|  | ii. | Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula. |
|  | iii. | Compute **one** iterations to estimate the solutions to 2 d.p. with |
|  | iv. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **8.** | Given the linear system: | |
|  | i. | Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p. |
|  | ii. | Reduce the above system to diagonally dominant form and write the corresponding |
|  | iii. | Gauss-Seidel iteration formula. |
|  | iv. | Compute one iterations to estimate the solutions to 2 d.p. with |
|  |  |  |
| **9.** | Tracy, Danielle and Sherri bought snacks for a birthday party. They each bought the items shown in the following table at the local convenience store: | |
|  |  | |
|  | |  |  |  |  | | --- | --- | --- | --- | | Number of bags of potato chips | Number of litres of pop | Number of chocolate bars | Cost  ($) | | 2 | 9 | 5 | 21.00 | | 3 | 2 | 10 | 20.88 | | 8 | 3 | 4 | 13.17 | | |
|  |  | |
|  | i. | Construct the system of linear equation from the above problem. |
|  | ii. | Reduce the system obtain in (i) to diagonally dominant form. |
|  | iii. | Write the corresponding Gauss-Seidel iteration formula. |
|  | iv. | Compute two iterations to estimate the roots to 2 d.p using the following initial values  . |
|  | v. | Justify your result by direct substitution in the original equations. |
|  | vi. | Write MATLAB codes to iterate the above formula four times. |
|  |  |  |
| **10.** | Jesse, Maria and Charles went to the local craft store to purchase supplies for making decorations for the upcoming dance at the high school. Jesse purchased three sheets of craft paper, four boxes of markers and five glue sticks. His bill, before taxes was $24.40. Maria spent $30.40 when she bought six sheets of craft paper, five boxes of markers and two glue sticks. Charles, purchases totaled $13.40 when he bought three sheets of craft paper, two boxes of markers and one glue stick. Determine the unit cost of each item. | |
|  | i. | Construct the system of linear equation from the above problem. |
|  | ii. | Solve the system of linear equation using Gaussian elimination with partial pivoting. |
|  | iii. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **11.** | A local computer company sells three types of laptop computers to three nearby stores. The number of laptops ordered by each store and the amount owing to the company for the order is shown in the following table: | |
|  |  |  |
|  | |  |  |  |  |  | | --- | --- | --- | --- | --- | | Store | Laptop A | Laptop B | Laptop C | Amount Owing($) | | Wal-Mart | 10 | 8 | 6 | 21 200 | | Sears | 7 | 9 | 5 | 18 700 | | Target | 8 | 4 | 3 | 13 000 | | |
|  |  | |
|  | i. | Write a system of equations to represent the above information. |
|  | ii. | Solve the system of linear equation using Gaussian elimination with partial pivoting. |
|  | iii. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **12.** | Consider the linear system: | |
|  | i. | Reduce the above system to diagonally dominant form. |
|  | ii. | Write the corresponding Gauss-Seidel iteration formula. |
|  | iii. | Compute two iterations to estimate the roots to 3 d.p. |
|  | iv. | Justify your result by direct substitution in the original equations. |
|  | v. | Write MATLAB codes to iterate the above formula four times.  Answer: |
|  |  |  |
| **13.** | The following system of equations was generated by applying the mesh current law in the  circuit shown in the adjacent figure.  2 Ω  **20 V**  40 Ω  35 Ω  **I3**  30 Ω  **I2**  **I1**  45 Ω | |
|  | i. | Write down the system in simplified form. |
|  | ii. | Solve the system of equations using Gauss-Seidel iteration correct to 2 decimal places. |
|  |  |  |
| **14.** | Given the linear system: | |
|  | i. | Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p. |
|  | ii. | Reduce the above system to diagonally dominant form and write the corresponding Gauss- Seidel iteration formula. |
|  | iii. | Compute **one** iterations to estimate the solutions to 2 d.p. with |
|  | iv. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  |  |
| **15.** | Given the linear system: | |
|  | i. | Use Gaussian elimination with partial pivoting to solve the system giving results to 2 d.p. |
|  | ii. | Reduce the above system to diagonally dominant form and write the corresponding Gauss-Seidel iteration formula. |
|  | iii. | Compute one iterations to estimate the solutions to 2 d.p. with |
|  | iv. | Write MATLAB codes to solve by left division (backslash) operator. |
|  |  | Answer: |
|  |  |  |
|  |  |  |
| **18.** |  |  |